Chapter 1: Intro to Diff. Equations

Some terminology

- A *differential equation* is any equation involving a derivative.

- An *ordinary differential equation (ODE)* is an equation involving derivatives relating only two variables (dependent/independent).

- A *partial differential equation (PDE)* is an equation involving partial derivatives.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

We only study ODE's in this course.

- The *order* of an ODE is the highest derivative that appears in the equation.

$$\frac{dy}{dt} - y + \sin(t) = \frac{d^2y}{dt^2}$$

second order

ONLY t & y

$$\frac{dP}{dt} = kP$$

1st order

constant

$$\frac{dy}{dt} = t^2$$

independent variable

we know how to solve

$$\frac{dy}{dt} = y + t$$

dependent variable?

Checking Solutions *(like 4-6 in HW)*

An explicit *solution* to a differential equation is a function that satisfies the equation.

FIRST ORDER

dependent variable

Example 1: $y' - 4y = 0$ has one solution that looks like $y(t) = e^{rt}$. Find *r*.

$y' = re^{rt}$

$\star \quad \underbrace{y'}_{re^{rt}} - \underbrace{4y}_{4e^{rt}} = 0$

$re^{rt} - 4e^{rt} \overset{?}{=} 0 \quad \Big\{$ FOR ALL t!

$e^{rt}(r-4) = 0$

$\Rightarrow \quad r = 4$

$\boxed{y = e^{4t} \text{ is a sol'n}}$

Example 2: $y'' - \frac{6}{t^2} y = 0$ has two solutions that look like $y(t) = t^r$. Find the two values of *r*.

2nd order

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

$\star \quad \underbrace{y''}_{r(r-1)t^{r-2}} - \frac{6}{t^2}\underbrace{y}_{t^r} = 0$

$r(r-1)t^{r-2} - \frac{6}{t^2} t^r \overset{?}{=} 0 \quad \Big\{$ FOR ALL t!

$t^{r-2}(r^2 - r - 6) = 0$

$t^{r-2}(r-3)(r+2) = 0$

$r = 3$ or $r = -2$

$\boxed{\begin{array}{l} y_1(t) = t^3 \text{ is a sol'n} \\ \hline y_2(t) = t^{-2} \text{ is a sol'n} \end{array}}$

2nd order, two very different sol'ns, hum?!

Example 3: $y'' - y' = 3x$ has one solution that look like

$y(t) = e^x + rx + sx^2$. Find *r* and s.

$y' = e^x + r + 2sx$

$y'' = e^x + 0 + 2s$

$$\boxed{y(t) = e^x + 3x - \frac{3}{2}x^2}$$

$$\underbrace{y''}_{\cancel{e^x} + 2s} - \underbrace{y'}_{(\cancel{e^x} + r + 2sx)} = 3x$$

$$\overset{?}{=} 3x \quad \text{(for all } x\text{)}$$

$$2s - r - 2sx = 3x$$

$$\underbrace{2s - r}_{\text{CONSTANT TERM}} - \underbrace{2s}_{\text{COEFFICIENT OF } x} x = \underbrace{3}_{\text{COEF. OF } x} x \quad \underbrace{}_{\text{CONSTANT TERM IS ZERO!}}$$

So $-2s = 3 \Rightarrow \boxed{s = -\frac{3}{2}}$

AND

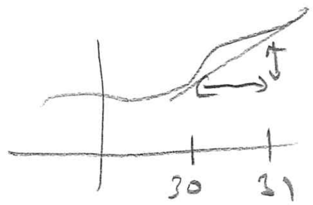
$2s - r = 0 \Rightarrow -3 - r = 0 \Rightarrow \boxed{r = 3}$

Applications/Units (7-10 in HW)

$$\frac{dy}{dt} =$$

"rate of change of y with respect to t"

Example:



Let $P(t)$ ="population after t years". Assume the population grows at a rate proportional to its size. That is,

$$\underbrace{\frac{dP}{dt}}_{\frac{\text{people}}{\text{year}}} = \underbrace{kP}_{\text{people added to population each year}}$$

For example, $K = 0.1$ (10%)
$\underbrace{K = 0.1}_{\text{relative growth rate.}}$

$\boxed{\text{See handout}}$

For example, what if there are 600 people right now, then $KP = 0.1 \cdot 600 = 60$ people

So rate added is 60 people/yr at this moment.

And if there are 7000 people at this moment then rate is $0.1P = 700$ people/yr at this moment.

Units Discussion:

dependent variable

What if P is in thousands of people?

Such as $P_2(t) = \frac{P(t)}{1000}$ thousand people. What is $\frac{dP_2}{dt}$?

$$\frac{dP}{dt} = 0.1P$$

$$P_2 = \frac{1}{1000} P$$

diff. both sides

$$\frac{dP_2}{dt} = \frac{1}{1000}\frac{dP}{dt}$$

$$\Rightarrow \frac{dP_2}{dt} = \frac{1}{1000} 0.1P$$

AND $\underbrace{P = 1000 P_2}$

$$\Rightarrow \frac{dP_2}{dt} = \frac{1}{\cancel{1000}} 0.1 \cdot \cancel{1000} P_2$$

$$\frac{dP_2}{dt} = 0.1 P_2$$

~~INDEPENDENT VARIABLE~~ BOTH SIDES SCALE ⇒ NO CHANGE TO DIFF. EQ.

~~this~~ of HW

Independent variable

What if *t* is in days instead of years?

Such at $t_2 = 365t$ days.

What is $\frac{dP}{dt_2}$?

$$\frac{dP}{dt} = \underbrace{0.1P}\quad \frac{\text{people}}{\text{year}}\quad \frac{1\text{ year}}{365\text{ days}}$$

☆ NEED CHAIN RULE

$$\frac{dP}{dt_2} = \frac{dP}{dt}\frac{dt}{dt_2}$$

$$t = \frac{1}{365} t_2$$

$$\Rightarrow \frac{dt}{dt_2} = \frac{1}{365}$$

$$\frac{dP}{dt_2} = \frac{dP}{dt}\frac{1}{365}$$

$$\boxed{\frac{dP}{dt_2} = \frac{1}{365} 0.1P \quad \frac{\text{people}}{\text{day}}}$$

☆ OR DIMENSIONAL ANALYSIS

Some Motivation

Air Resistance Example:

Assume an object with mass m is dropped from 1000 meters with an initial velocity of $v(0) = 0$ m/s.

Recall: Newton's Law

$$ma = F \quad (Force)$$

If we write $a(t) = v'(t) = h''(t)$, then we see this leads to an ODE.

1. No air resistance, then the only force is gravity F = -mg ($g = 9.8\ m/s^2$).

$$mv' = F_g = -mg$$

Initial ***V****alue* ***P****roblem* (IVP)

$$v' = -g$$
$$v(0) = 0$$

$$v' = -9.8$$

$$v(t) = -9.8t + C$$

$$v(0) = 0 \Rightarrow -9.8(0) + C = 0$$
$$\Rightarrow C = 0$$

$$\boxed{v(t) = -9.8t}$$

$$h(t) = -4.9t^2 + D$$

$$h(0) = 1000 \Rightarrow D = 1000$$

2. Assume air resistance exerts a force proportional to speed in the opposite direction.

$$mv' = F_g + F_A = -mg - rv$$

So

$$v' = -g - \frac{r}{m}v$$

independent variable

$$v(0) = 0$$

CAN'T INTEGRATE BOTH SIDES!!!

???!

Sol'n looks like

$$v(t) = A + Be^{ct}$$

$$v' = Bce^{ct}$$

TRY TO SOLVE

WHAT VIDEO ON MY WEBSITE OF SKYDIVER!

HOW???

WE DON'T KNOW YET.

NEED MORE METHODS.

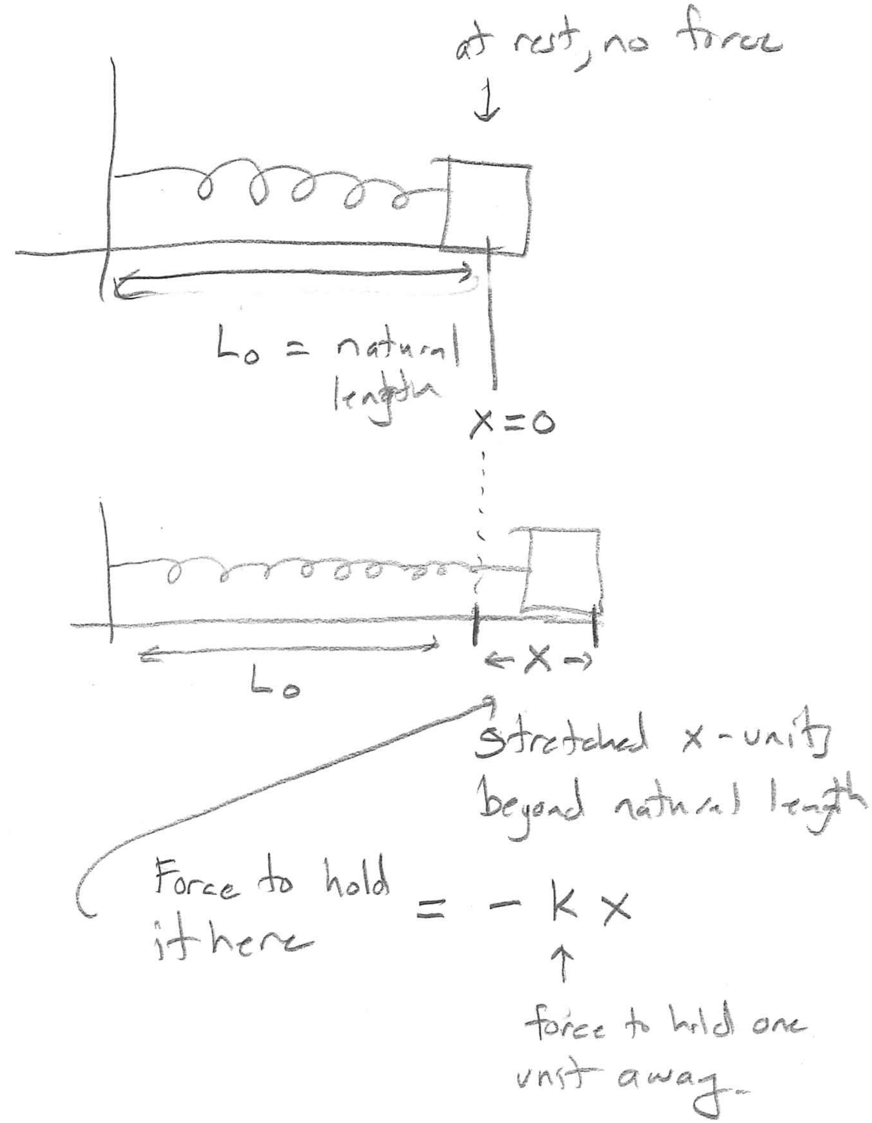
Mass-Spring Example:

Assume an object with mass m is attached to a spring that is attached to a wall. Natural length is the distance from the wall at which the mass is at rest (no stretch or push force).

Let x = "the distance the spring is stretched beyond natural length".

Hooke's Law says force due to the spring is proportional to x and in the opposite direction. In other words,

$$Force = -kx$$



So, once again, if you want to model the motion of the mass after you stretch it and let go, you use Newton's Law:

$$ma = F = -kx$$

and $a(t) = x''(t)$

So

$$m\,x'' = -kx$$

It turns out that one solution to this is $x(t) = cos(\omega t)$

Aside: This is called a "simple harmonic oscillator" and ω is the "natural frequency" (radians/time). And $2\pi/\omega$ is the wavelength (time from peak-to-peak)

FIND ω

$x' = -\omega \sin(\omega t)$

$x'' = -\omega^2 \cos(\omega t)$

$\Rightarrow -m\omega^2 \cos(\omega t) = -k\cos(\omega t)$

$\Rightarrow 0 = (m\omega^2 - k)\cos(\omega t)$ FOR ALL t!

$\Rightarrow m\omega^2 - k = 0$

$\omega^2 = \frac{k}{m}$

$\boxed{\omega = \pm\sqrt{\frac{k}{m}}}$ natural frequency

$y(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right)$

Also $\sin(\omega t)$ is also a solution

Circuits Example: Kirchoff's laws observe that the sum of the voltage drops in a circuit equals zero (source, resistance, capacitance, inductance).

So $-V_0 + V_R + V_C + V_L = 0$

with

$$I = \frac{dq}{dt} = q'$$

$$V_R = RI = Rq'$$

$$V_C = \frac{q(t)}{C} = \frac{q}{C}$$

$$V_L = L\frac{dI}{dt} = Lq''$$

So $V_0 = V_R + V_C + V_L$

$$V_0 = Rq' + \frac{q}{C} + Lq''$$

HINTS!

$\boxed{\text{IN HOMEWORK}}$, $V_L = 0$ (no inductor)

part (a)

AND $V_0 = V$

So $V = Rq' + \frac{q}{C}$

Solve for q'!

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part (b)

AND

$V_C(t) = \frac{q(t)}{C}$

So $\frac{dV_C}{dt} = \frac{1}{C}\frac{dq}{dt}$